Effect of symmetry breaking on level curvature distributions

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We derive an exact general formalism that expresses the eigenvector and the eigenvalue dynamics as a set of coupled equations of motion in terms of the matrix elements dynamics. Combined with an appropriate model Hamiltonian, these equations are used to investigate the effect of the presence of a discrete symmetry in the level curvature distribution. An explanation of the unexpected behavior of the data regarding frequencies of acoustic vibrations of quartz block is provided.

I. INTRODUCTION

The usefulness of the study of statistical properties of eigenvalues and eigenvectors of quantum systems has already been demonstrated in many areas of physics. A lot can be learned, especially about symmetries, by just employing the appropriate statistics. It has also become clear that these statistics follow universal patterns that can be modeled by probability distributions extracted from an ensemble of random Hamiltonians of the same class of the underlying symmetry of the system under study [1]. This has been a field of intense investigation over the last 2 decades [2]. These activities have concentrated their effort on what we can call the "statics" of the problem, in which stationary Hamiltonians are considered. More recently, however, interest has also been directed to the dynamical aspects of the same question.

The "dynamics" consists in considering a given Hamiltonian as a function of a parameter (representing "time"). The statistical properties that characterize the evolution are then studied as the parameter is varied. Only evolutions that preserve the symmetry class of the Hamiltonian are considered. Several measures have been introduced to investigate this kind of evolution. One of the most frequently used ones is the probability distribution of the level curvature, which can be thought of as "acceleration" as it is defined in terms of the second derivative with respect to the parameter. These distributions measure correlations among the set of eigenvalues. Another measure that is commonly used is the two-point correlation function between first derivatives ("velocities"). Given some generic level, this two-point correlation function is obtained by calculating the "velocity" at two different values of the parameter [3]. Measures have also been considered to characterize the evolution of the eigenvectors [4].

These studies started with Wilkinson's pioneering work that investigated the dependence of the eigenvalues of a fully chaotic billiard as a function of its shape [5]. The plot of the trajectories of levels as a function of the parameter that controls the shape, exhibits a typical pattern of avoided crossings. A measure of these is provided by the curvature of the trajectory. There is now an analytical evidence that, in the fully chaotic regime, the curvatures, after an appropriate rescaling, follow a universal simple distribution. The tail of this distribution has been investigated, and an asymptotic dependence inversely proportional to the third power of the curva-

ture was established, for fully chaotic systems that are time-reversal invariant and thus governed by the Gaussian orthogonal ensemble (GOE) [6]. The expression

$$P(k) = \frac{1}{2(1+k^2)^{3/2}} \tag{1}$$

was then proposed for the entire domain of the curvature k. Finally, it has been proved that this function gives the exact distribution of curvatures, in the case of random matrix ensembles [7]. The power $\frac{3}{2}$ in the denominator is the GOE value of $(\beta+2)/2$ with $\beta=1,2$, and 4 for the GOE, Gaussian unitary ensemble, and Gaussian symplectic ensemble, respectively.

Recently, the difficult task of experimentally checking this prediction was undertaken by Bertelsen *et al.* at the Center for Chaos and Turbulence of the Niels Bohr Institute [8]. They studied the dependence on the temperature, the external parameter in the system, of the spectrum of frequencies of quartz blocks. In previous investigations [9], they have found that the spectra of frequencies of quartz blocks obey statistical models based on random matrix theories. The dynamics of the frequencies, as a function of the temperature, was therefore measured for a quartz block whose static statistical properties were previously established. However, the data obtained have shown a deviation from the above expected curvature distribution. This deviation, although slight, is significant and not yet completely understood. We are going to show, in the present paper, that the data can in fact be fully understood if one requires the average curvature to be equal to 1, as is implied by the universal distribution, Eq. (1).

So far, all studies of parametric correlations have been concentrated on the fully chaotic regime when the system statistics are well described by the Gaussian ensembles of random matrix theory, in particular, the GOE, if there is time-reversal invariance. The partially chaotic situation has been little investigated. We intend here to provide a systematic discussion of this situation. In Sec. II, we develop the formalism and the model we employ, and in Sec. III, we present the numerical results and the discussion. We verify that, at the GOE limit, the above universal expression for the level curvature distribution is obtained. As some degree of symmetry is introduced, it is found that the distribution becomes narrower. However, as the symmetry is progressively

introduced, the distribution returns to the universal function, if the average of curvatures is imposed as 1.

II. THE FORMALISM AND THE MODEL

We shall now derive a set of equations that describe simultaneously the dynamics of the energy levels and of the eigenvector components of a Hamiltonian H. These equations contain the equations of motion of the matrix elements of H, whose dependence on the parameter t, representing the "time," is supposed to be given. Our starting point is the general matrix equation

$$H = U H_D U^\dagger, \tag{2}$$

where H is an $N \times N$ real symmetric matrix, H_D is the diagonal matrix constructed with the N eigenvalues, and U is the unitary matrix whose columns are the N eigenvectors. Assuming that H_D and U also depend on the parameter t, differentiating Eq. (1) with respect to t we get

$$\dot{H} = U \dot{H}_D U^\dagger + \dot{U} H_D U^\dagger + U H_D \dot{U}^\dagger, \tag{3}$$

where the derivative is indicated by a dot. Multiplying Eq. (3) by $U^\dagger$ from the left, by U from the right, and defining the matrix $S = U^\dagger \dot{U} = -\dot{U}^\dagger U$ we obtain the equation of motion

$$\dot{H}_D = [H_D, S] + P, \tag{4}$$

where the matrix $P = U^\dagger \dot{H} U$ was introduced. On the other hand, we find for P the conjugate evolution equation

$$\dot{P} = [P, S] + U^\dagger \ddot{H} U. \tag{5}$$

By choosing a particular model, i.e., the dependence of the matrix element on the parameter t, these equations can be employed in several contexts. They can be used, for example, to construct an alternative method of matrix diagonalization, or, by requiring the matrix elements to satisfy appropriate Langevin equations, they lead to Dyson's Brownian motion model [10]. Here, we concentrate on the simple model given by

$$H = H_1 \cos t + H_2 \sin t, \tag{6}$$

where H_1 and H_2 are a couple of fixed, i.e., parameter independent, random matrices taken from the same matrix ensemble, and t is the parameter. If in Eq. (6) H_1 and H_2 are taken from a Gaussian ensemble, the evolution will preserve the probability distribution, so that H will remain in the same ensemble. With this choice, Eq. (5) becomes

$$\dot{P} = [P, S] - H_D. \tag{7}$$

The pair of coupled equations, Eqs. (4) and (7), have the explicit solution

$$H_D(t) = U^\dagger(t)[H_D(0)\cos t + P(0)\sin t]U(t)$$

and

$$P(t) = U^\dagger(t)[-H_D(0)\sin t + P(0)\cos t]U(t),$$

where $U(t)$ is the solution of the equation $\dot{U} = US$ given by

$$U(t) = T \exp \int_0^t S(\tau) d\tau$$

with T being the time-ordering operator.

To implement this solution numerically a basis has to be chosen to express the eigencomponents. Since our objective is to investigate the curvature, a quantity related to the behavior of the eigenvalues as the external parameter is varied, it is convenient to use the instantaneous Hamiltonian eigenstates as basis vectors. In this case, from the diagonal part of Eq. (4), we derive

$$\dot{E}_k = P_{kk} \tag{8}$$

and using in Eq. (5) the relation $S_{kl} = P_{kl}/(E_k - E_l)$ [obtained from the off-diagonal part of Eq. (4)] we can then derive the equations

$$\dot{P}_{kl} = -\frac{P_{kk} - P_{kl}}{E_k - E_l} P_{kl} + \sum_{m=1, m \neq k, l}^{N} P_{km} P_{lm} \left(\frac{1}{E_k - E_m} \right.$$
$$\left. + \frac{1}{E_l - E_m} \right) \tag{9}$$

and

$$\dot{P}_{kk} = -E_k + \sum_{m=1, m \neq k}^{N} \frac{2 P_{km}^2}{E_k - E_m}. \tag{10}$$

This set of coupled equations is one of the main results of this paper. All calculations will be based on it. Thus the "accelerations," i.e., the levels' curvature, are just given by Eq. (10).

Regarding the random matrix ensemble, we shall work with a Gaussian ensemble that interpolates between one GOE and two decoupled GOE's. This ensemble has been already employed with a very satisfactory result in the analysis of data relative to symmetry breaking [11,12] in nuclear [13] and acoustic systems [9]. It can be defined by the following operator equation [14]:

$$H = P H^{GOE} P + Q H^{GOE} Q + \lambda (P H^{GOE} Q + Q H^{GOE} P), \tag{11}$$

where $P = \sum_{i=1}^{M} P_i$, $Q = 1 - P$, and $P_i = |i\rangle\langle i|$, $i = 1, \ldots, N$ are projection operators, $0 \leq \lambda \leq 1$ is the parameter that controls the transition, and H^{GOE} denotes a GOE matrix whose elements follow a joint probability distribution given by

$$P(H^{GOE}) \propto \exp[-\alpha \operatorname{tr}(H^{GOE})^2], \tag{12}$$

with α being an arbitrary scaling parameter. With the above definitions, $\lambda = 1$ corresponds to the GOE case, while $\lambda = 0$ corresponds to block diagonal random matrices, made up of two GOE matrices of sizes $M \times M$ and $(N-M)(N-M)$.
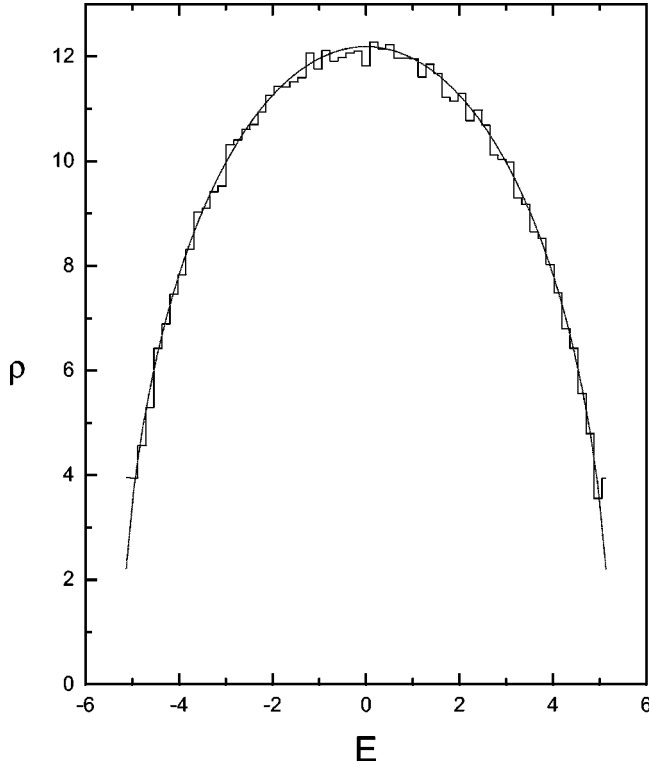
FIG. 1. Densitiy of levels: comparison of the calculated density (histogram) with the semicircle law (15) (solid line). The calculation corresponds to matrices of dimension $N=100$, and $\varepsilon=0.32$. Both density (ρ) and energy (E) are in arbitrary units.

Regarding this ensemble, it is important to stress that λ is not a more convenient parameter to work with, since the transition is also dependent on the matrix size N. Independence on the dimension is obtained by introducing the scaled parameter

$$\varepsilon = \sqrt{N}\lambda. \tag{13}$$

III. NUMERICAL RESULTS AND DISCUSSION

Before presenting the results, we discuss the rescaling variables necessary to extract a universal behavior. First, we have to unfold the spectrum, that is, we work with a new spectrum generated by the transformation

$$x_l = \int_{-\infty}^{E_l} dE \bar{\rho}(E) \quad \text{for} \quad l=1,\ldots,N, \tag{14}$$

where $\bar{\rho}(E)$ is the averaged level density. Without loss of generality, we consider in the calculation only the symmetric situation, $N=2M$, in which the matrices are decomposed into blocks of equal size. In this case, the average density is given by the Wigner's semicircle law [15]. With an appropriate scaling that guarantees the correct value of the second moment of the eigenvalue, the level density is given by

$$\bar{\rho}(E) = \frac{4\alpha}{\pi(1+\lambda^2)} \sqrt{\frac{N}{2\alpha}(1+\lambda^2) - E^2}. \tag{15}$$
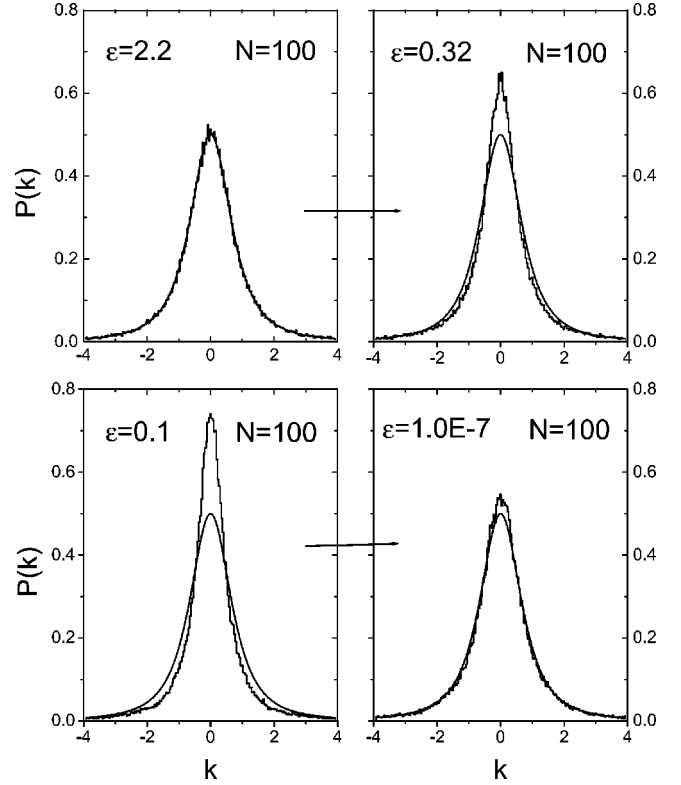

FIG. 2. Level curvature distributions: comparison of the calculated histograms with the theoretical prediction (1) (solid line). The calculations correspond to matrices of dimension $N=100$, and for the values of ε indicated in the figure. The variables $P(k)$ and k are dimensionless.

In Fig. 1, we show the nice fit obtained with this expression when compared with the numerical values of $\bar{\rho}(E)$ generated within the two coupled GOE's ensemble alluded to above.

Second, we need some normalization of the accelerations. This is a controversial issue that requires some discussion. On one hand, it has been proposed that the parameter t should be replaced by a new dimensionless parameter τ related to t by [16]

$$\frac{d\tau}{dt} = \sqrt{\langle \dot{x}^2 \rangle}, \tag{16}$$

where the average of the velocity is made over the whole set of eigenvalues or, equivalently, over the ensemble. The level curvature is then defined in terms of these new scaled variables as

$$K = \frac{1}{\pi} \frac{d^2 x}{d\tau^2} = \frac{1}{\pi \langle \dot{x}^2 \rangle} \left(\ddot{x} - \frac{\langle \dot{x}\ddot{x} \rangle}{\langle \dot{x}^2 \rangle} \dot{x} \right), \tag{17}$$

where $\dot{x} = \bar{\rho}(E)\dot{E}$ and $\ddot{x} = \bar{\rho}(E)\ddot{E} + d\bar{\rho}(E)/dE(\dot{E})^2$.

On the other hand, the universal curvature distribution, Eq. (1), implies that $\langle |k| \rangle = 1$. It is not at all clear that the scaled curvatures given by Eq. (17) will satisfy this condition. Thus, we imposed the normalization
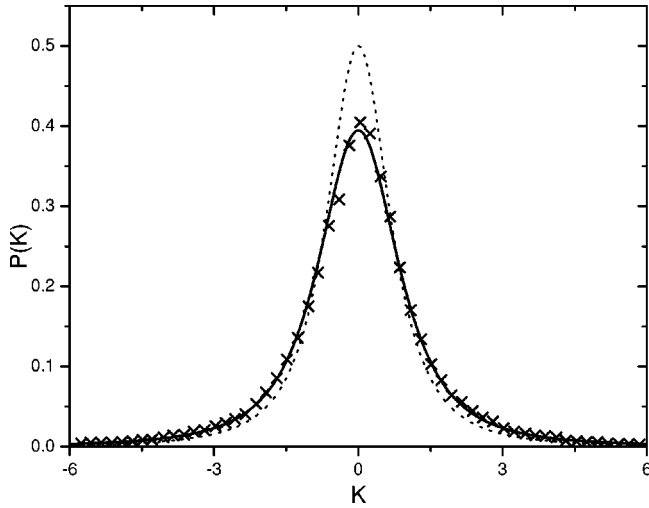
FIG. 3. Fitting of the data (crosses) of Ref. [8] with the parametrized distribution (20) (solid line). The best fit ($\chi^2 = 0.000\,04$) was obtained with $\gamma = 1.27 \pm 0.01$. The dotted curve corresponds to the universal distribution. The variables are dimensionless.

$$k = \frac{K}{\langle |K| \rangle} \tag{18}$$

with K given by Eq. (17). Our calculations have shown that this last step is necessary in order to get stable results, i.e., independent of the subset of levels of the spectra over which the statistics is performed.

The behavior of the distribution, Eq. (1), for large curvatures can be traced to the level spacing distribution. In fact, large curvatures can be considered, approximately, as inversely proportional to the small level spacing s. Thus if we assume $s \propto 1/k$ and use the fact that, in the GOE case, $P(s)$ is linear in s, we obtain

$$P(k) \sim P(s) \left| \frac{ds}{dk} \right| \sim k^{-3}, \tag{19}$$

as predicted by Eq. (1). As a consequence, as symmetry is introduced by decreasing the parameter λ, one would expect

a reduction on the probability of large curvatures with the distribution becoming narrower. We shall see that this indeed happens.

Our main results are presented in Fig. 2, where the curvature distributions were calculated for four values of the scaled parameter (13). The figures show that, at the two extreme situations, namely, in the one GOE limit and in the two fully decoupled GOE's limit, the curvatures distribute themselves according to the universal distribution. We would expect this kind of behavior in the latter limit, since the levels in each block become completely independent of the levels in the other. Thus, their trajectories can cross freely. As the above discussion predicted, the distributions are narrower in the intermediate region. We stress that these results are strongly dependent on the renormalization (18).

Turning now to the question of the behavior (distributions wider than the universal) presented by the data of Ref. [8], one possible explanation would be that the curvatures do not average to 1. To check this point we have fitted the data with the distribution

$$P(K) = \frac{1}{2\gamma[1 + (K/\gamma)^2]^{3/2}} \tag{20}$$

in which the average curvature $\gamma = \langle |K| \rangle$ is treated as a free parameter. The best fit, obtained with $\gamma = 1.27 \pm 0.01$, is displayed in Fig. 3. This excellent fit makes this explanation plausible.

In conclusion, we have investigated the effect of the symmetry breaking on the level curvature distribution using a random matrix ensemble that allows for a transition from one GOE to two decoupled GOE's. We have also provided an explanation for the discrepancy [8] of the data regarding the temperature dependence of frequencies of acoustic vibrations of quartz blocks.

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